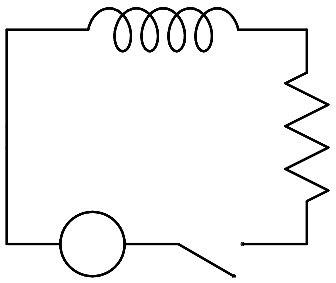
II. Example 2: R-L DC Circuit

Physical characteristics of the circuit: 100 volt DC battery connected in series with a 2 henry inductor and a 6 ohm resistor; current flows when the open switch is closed.



Questions:

[a] Describe in words how the current changes over time.

[b] What is the current 1 second after the switch is closed?

[c] At what time does the current equal 8 amps?

[d] At what time does the current equal k amps, where k is a positive constant?

Solution of Circuit IVP:

By Kirchhoff's laws we have $E_L + E_R = EMF$ which, with $E_L = L \cdot I'(t)$ and $E_R = R \cdot I(t)$, translates into the following Initial Value Problem (for $t \geq 0$):

$$2\,I'(t) + 6\,I(t) = 100, \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

We can solve for I using a method called *separation of variables*. First, we will divide through by 2, replace $I(t)$ by I, and use the differential notation for derivatives:

$$\frac{dI}{dt} + 3I = 50$$

Solving for $\frac{dI}{dt}$ shows that this is not a simple integration problem because the derivative of I depends on both t and I

$$\frac{dI}{dt} = 50 - 3I \tag{$*$}$$

Outline of solution by *separation of variables*

Use algebra to rewrite $(*)$ as

$$\frac{dI}{50 - 3I} = dt$$

and integrate both sides to obtain

$$-\frac{1}{3}\ln|50-3I| = t + C$$

which with the initial condition $I(0)=0$ yields the circuit current

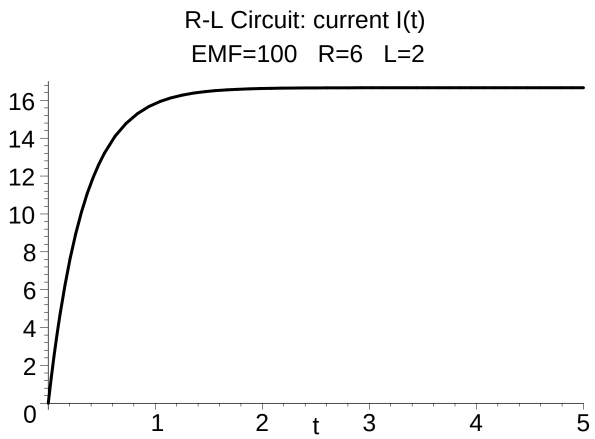
$$I(t) = \frac{50}{3}\left(1-e^{-3t}\right), \quad t \geq 0$$

More details for all these steps may be found below, after the Answers.

Answers:

[a] Describe in words how the current changes over time.

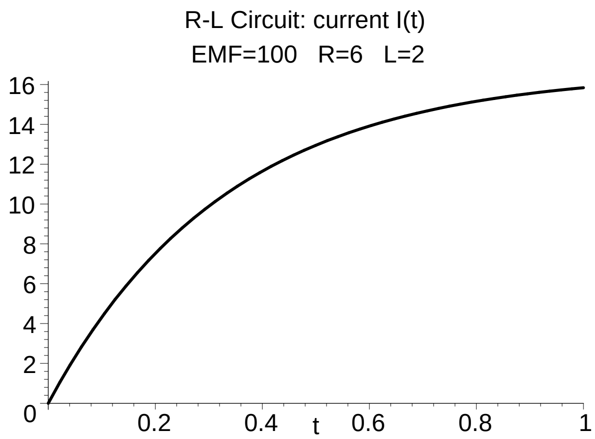
The following graph shows how $I(t)$ increases from 0 at $t=0$ toward an asymptotic limit $50/3$ as t increases. This asymptotic limit is called the *steady-state* current.



[b] What is the current 1 second after the switch is closed?

$$I(1) = \frac{50}{3}\left(1-e^{-3}\right) \approx 15.84$$

which agrees with the following graph.



[c] At what time does the current equal 8 amps?

Solve

$$\frac{50}{3}\left(1 - e^{-3 \cdot t}\right) = 8$$

to get $t = -\frac{1}{3}\ln\left(\frac{13}{25}\right) \approx 0.22$, which agrees with the preceding graph.

[d] At what time does the current equal k amps, where k is a positive constant?

Using the same steps as in the preceding part

$$t = -\frac{1}{3}\ln\left(1 - \frac{3k}{50}\right)$$

for any $k < 50/3$.

Details of solution by *separation of variables*

After multiplying both sides of the ODE

$$\frac{dI}{dt} = 50 - 3I \tag{$*$}$$

by dt, we get the ODE in differential form

$$dI = (50 - 3I)\,dt$$

Divide both sides by $50 - 3I$ in order to *separate variables: put anything involving I on one side and anything involving t on the other side*:

$$\frac{dI}{50 - 3I} = dt \tag{1}$$

Now we are allowed to integrate each side separately and still have equality. The right side of equation (1) is easy:

$$\int dt = t + C$$

where C is an arbitrary constant. The left side of equation (1) looks related to the integral $\int \frac{1}{x}\,dx$. So we use the substitution

$$\begin{aligned} & x = 50 - 3I \\ \text{to get} \qquad & \frac{dx}{dI} = -3 \\ \text{or} \qquad & dI = -\frac{1}{3}dx \end{aligned}$$

Then in equation (1) we replace $50 - 3I$ with x and dI with $-\frac{1}{3}dx$ and integrate in order to get the left side to equal

$$\begin{aligned} \int \frac{1}{50 - 3I}\,dI &= \int \frac{1}{x}\left(-\frac{1}{3}dx\right) \\ &= -\frac{1}{3}\int \frac{1}{x}\,dx \\ &= -\frac{1}{3}\ln|x| + C \\ &= -\frac{1}{3}\ln|50 - 3I| + C \end{aligned}$$

Hence equation (1), after both sides are integrated, becomes (collecting all arbitrary constants on the right hand side as a single arbitrary constant)

$$-\frac{1}{3}\ln|50 - 3I| = t + C \tag{2}$$

Since there is no current when the switch is thrown, we let $I = 0$ when $t = 0$ to solve for C

$$-\frac{1}{3}\ln|50 - 0| = 0 + C \Longrightarrow C = -\frac{1}{3}\ln 50$$

and so equation (2) becomes

$$-\frac{1}{3}\ln|50 - 3I| = -\frac{1}{3}\ln 50 + t$$

It is usually preferable to solve for the dependent variable, I in this case. To do that, we first multiply both sides of the last equation by -3 to get

$$\ln|50 - 3I| = \ln 50 - 3t$$

then take the exponential (inverse logarithm) of both sides

$$e^{\ln|50-3I|} = e^{\ln 50 - 3t} \tag{3}$$

and then use a property of exponentials

$$e^{a+b} = e^a \times e^b$$

with $a = \ln 50$ and $b = -3t$ to get from equation (3)

$$\begin{aligned} |50 - 3I| &= e^{\ln 50} \times e^{-3t} \\ &= 50e^{-3t} \end{aligned}$$

since $e^{\ln 50} = 50$. Now $|x| = c \Longrightarrow x = \pm c$ so we have

$$50 - 3I = \pm 50e^{-3t}$$

Since we know that $I = 0$ at $t = 0$, we determine the sign to be $+$, allowing us to solve for I by dividing both sides of the last equation by 3 and then isolating I on one side

$$I(t) = \frac{50}{3}\left(1 - e^{-3t}\right), \quad t \geq 0$$